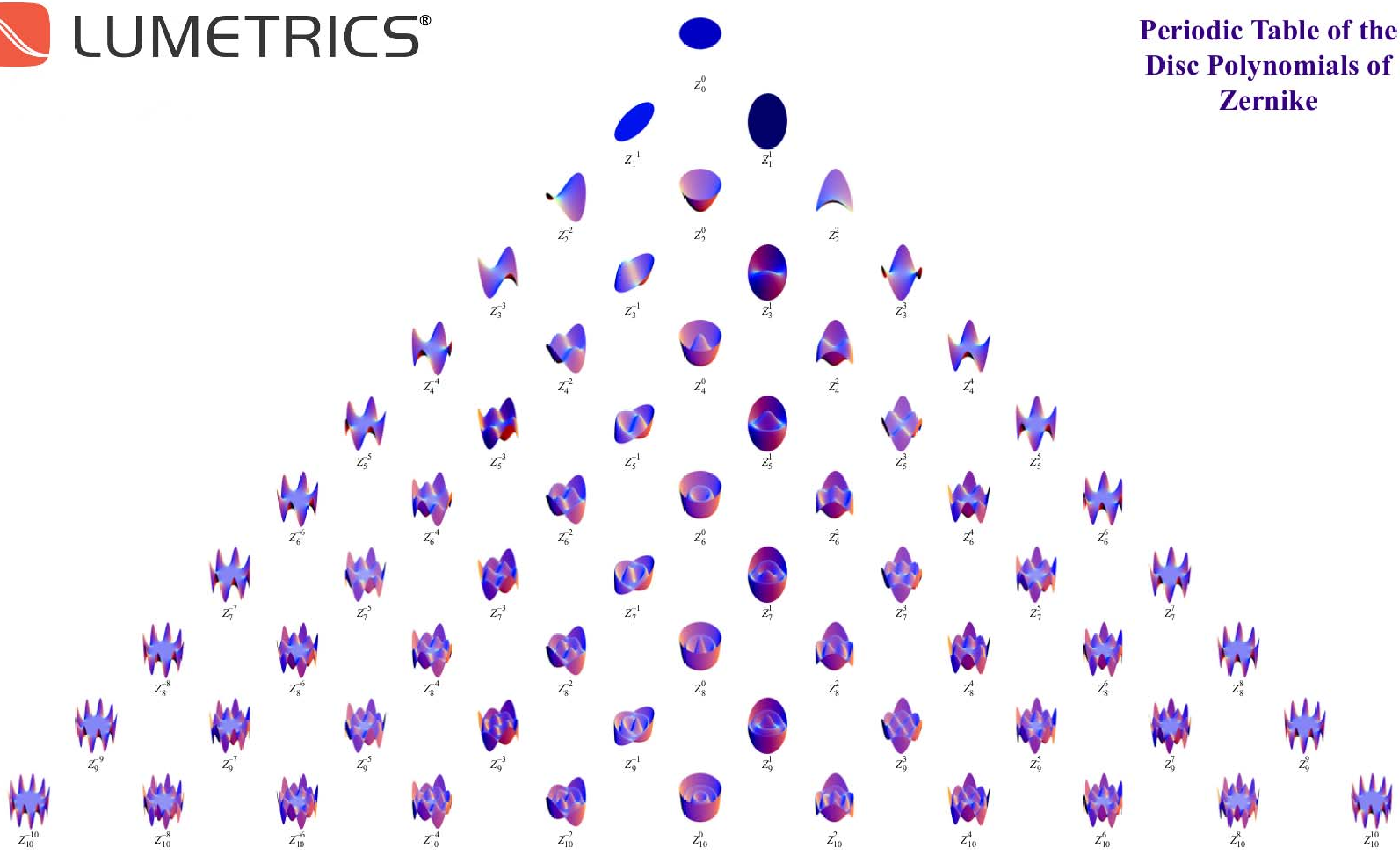
Periodic Table of the Disc Polynomials of Zernike



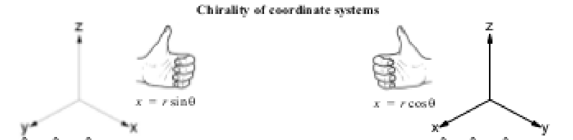
#	n	m	μ	Z_n^m	$Z_{n,\mu}$	Taylor	RMS	Polar form $(dx^2 \to r d\theta dr)$	Cartesian form $(dx^2 \to dx dy)$	Taylor coefficient vector
0	0	0	0	Z_0^0	$Z_{0,0}$	1	$\sqrt{2}$	1	1	((1))
1	1	–1	0	Z_1^{-1}	$Z_{1,0}$	x	2	$r\sin\theta$	x	((0),(1,0))
2	1	1	1	Z_1^1	$Z_{1,1}$	y	2	$r\cos\theta$	y	((0),(0,1))
3	2	–2	0	Z_2^{-2}	$Z_{2,0}$	x^2	$\sqrt{6}$	$r^2\sin 2\theta$	$2xy$	((0),(0),(0,2,0))
4	2	0	1	Z_2^0	$Z_{2,1}$	xy	$\sqrt{3}$	$2r^2-1$	$-1+2x^2+2y^2$	((–1),(0),(2,0,2))
5	2	2	2	Z_2^2	$Z_{2,2}$	y^2	$\sqrt{6}$	$r^2\cos 2\theta$	$-x^2+y^2$	((0),(0),(–1,0,1))
6	3	–3	0	Z_3^{-3}	$Z_{3,0}$	x^3	$2\sqrt{2}$	$r^3\sin 3\theta$	$-x^3+3xy^2$	((0),(0),(0),(–1,0,3,0))
7	3	–1	1	Z_3^{-1}	$Z_{3,1}$	x^2y	$2\sqrt{2}$	$(3r^3-2r)\sin\theta$	$-2x+3x^3+3xy^2$	((0),(–2,0),(0),(3,0,3,0))
8	3	1	2	Z_3^1	$Z_{3,2}$	xy^2	$2\sqrt{2}$	$(3r^3-2r)\cos\theta$	$-2y+3y^3+3x^2y$	((0),(0,–2),(0),(0,3,0,3))
9	3	3	3	Z_3^3	$Z_{3,3}$	y^3	$2\sqrt{2}$	$r^3\cos 3\theta$	y^3-3x^2y	((0),(0),(0),(0,–3,0,1))
10	4	–4	0	Z_4^{-4}	$Z_{4,0}$	x^4	$\sqrt{10}$	$r^4\sin 4\theta$	$-4x^3y+4xy^3$	((0),(0),(0),(0),(0,–4,0,4,0))
11	4	–2	1	Z_4^{-2}	$Z_{4,1}$	x^3y	$\sqrt{10}$	$(4r^4-3r^2)\sin 2\theta$	$-6xy+8x^3y+8xy^3$	((0),(0),(0,–6,0),(0),(0,8,0,8,0))
12	4	0	2	Z_4^0	$Z_{4,2}$	x^2y^2	$\sqrt{5}$	$6r^4-6r^2+1$	$1-6x^2-6y^2+6x^4+12x^2y^2+6y^4$	((1),(0),(–6,0,–6),(0),(6,0,12,0,6))
13	4	2	3	Z_4^2	$Z_{4,3}$	xy^3	$\sqrt{10}$	$(4r^4-3r^2)\cos 2\theta$	$3x^2-3y^2-4x^4+4y^4$	((0),(0),(3,0,–3),(0),(–4,0,0,0,4))
14	4	4	4	Z_4^4	$Z_{4,4}$	y^4	$\sqrt{10}$	$r^4\cos 4\theta$	$x^4-6x^2y^2+y^4$	((0),(0),(0),(0),(1,0,–6,0,1))
15	5	–5	0	Z_5^{-5}	$Z_{5,0}$	x^5	$2\sqrt{3}$	$r^5\sin 5\theta$	$x^5-10x^3y^2+5xy^4$	
16	5	–3	1	Z_5^{-3}	$Z_{5,1}$	x^4y	$2\sqrt{3}$	$(5r^5-4r^3)\sin 3\theta$	$4x^3-12xy^2-5x^5+10x^3y^2+15xy^4$	
17	5	–1	2	Z_5^{-1}	$Z_{5,2}$	x^3y^2	$2\sqrt{3}$	$(10r^5-12r^3+3r)\sin\theta$	$3x-12x^3-12xy^2+10x^5+20x^3y^2+10xy^4$	
18	5	1	3	Z_5^1	$Z_{5,3}$	x^2y^3	$2\sqrt{3}$	$(10r^5-12r^3+3r)\cos\theta$	$3y-12y^3-12x^2y+10y^5+20x^2y^3+10x^4y$	
19	5	3	4	Z_5^3	$Z_{5,4}$	xy^4	$2\sqrt{3}$	$(5r^5-4r^3)\cos 3\theta$	$-4y^3+12x^2y+5y^5-10x^2y^3-15x^4y$	
20	5	5	5	Z_5^5	$Z_{5,5}$	y^5	$2\sqrt{3}$	$r^5\cos 5\theta$	$y^5-10x^2y^3+5x^4y$	
21	6	–6	0	Z_6^{-6}	$Z_{6,0}$	x^6	$\sqrt{14}$	$r^6\sin 6\theta$	$6x^5y-20x^3y^3+6xy^5$	
22	6	–4	1	Z_6^{-4}	$Z_{6,1}$	x^5y	$\sqrt{14}$	$(6r^6-5r^4)\sin 4\theta$	$20x^3y-20xy^3-24x^5y+24xy^5$	
23	6	–2	2	Z_6^{-2}	$Z_{6,2}$	x^4y^2	$\sqrt{14}$	$(15r^6-20r^4+6r^2)\sin 2\theta$	$12xy-40x^3y-40xy^3+30x^5y+60x^3y^3-30xy^5$	
24	6	0	3	Z_6^0	$Z_{6,3}$	x^3y^3	$\sqrt{7}$	$20r^6-30r^4+12r^2-1$	$-1+12x^2+12y^2-30x^4-60x^2y^2-30y^4+20x^6+60x^4y^2+60x^2y^4+20y^6$	
25	6	2	4	Z_6^2	$Z_{6,4}$	x^2y^4	$\sqrt{14}$	$(15r^6-20r^4+6r^2)\cos 2\theta$	$-6x^2+6y^2+20x^4-20y^4-15x^6-15x^4y^2+15x^2y^4+15y^6$	
26	6	4	5	Z_6^4	$Z_{6,5}$	xy^5	$\sqrt{14}$	$(6r^6-5r^4)\cos 4\theta$	$-5x^4+30x^2y^2-5y^4+6x^6-30x^4y^2-30x^2y^4+6y^6$	
27	6	6	6	Z_6^6	$Z_{6,6}$	y^6	$\sqrt{14}$	$r^6\cos 6\theta$	$-x^6+15x^4y^2-15x^2y^4+y^6$	
28	7	–7	0	Z_7^{-7}	$Z_{7,0}$	x^7	4	$r^7\sin 7\theta$	$-x^7+21x^5y^2-35x^3y^4+7xy^6$	
29	7	–5	1	Z_7^{-5}	$Z_{7,1}$	x^6y	4	$(7r^7-6r^5)\sin 5\theta$	$-6x^5+60x^3y^2-30xy^4+7x^7-63x^5y^2-35x^3y^4+35xy^6$	
30	7	–3	2	Z_7^{-3}	$Z_{7,2}$	x^5y^2	4	$(2r^7-30r^5+10r^3)\sin 3\theta$	$-10x^3+30xy^2+30x^5-60x^3y^2-90xy^4-21x^7+21x^5y^2+105x^3y^4+63xy^6$	
31	7	–1	3	Z_7^{-1}	$Z_{7,3}$	x^4y^3	4	$(35r^7-60r^5+30r^3-4r)\sin\theta$	$-4x+30x^3+30xy^2-60x^5-120x^3y^2-60xy^4+35x^7+105x^5y^2+105x^3y^4+35xy^6$	
32	7	1	4	Z_7^1	$Z_{7,4}$	x^3y^4	4	$(35r^7-60r^5+30r^3-4r)\cos\theta$	$-4y+30y^3+30x^2y-60y^5-120x^2y^3-60x^4y+35y^7+105x^2y^5+105x^4y^3+35x^6y$	
33	7	3	5	Z_7^3	$Z_{7,5}$	x^2y^5	4	$(2r^7-30r^5+10r^3)\cos 3\theta$	$10y^3-30x^2y-30y^5+60x^2y^3+90x^4y+21y^7-21x^2y^5-105x^4y^3+63x^6y$	
34	7	5	6	Z_7^5	$Z_{7,6}$	xy^6	4	$(7r^7-6r^5)\cos 5\theta$	$-6y^5+60x^2y^3-30x^4y+7y^7-63x^2y^5-35x^4y^3+35x^6y$	
35	7	7	7	Z_7^7	$Z_{7,7}$	y^7	4	$r^7\cos 7\theta$	$y^7-21x^2y^5+35x^4y^3-7x^6y$	
36	8	–8	0	Z_8^{-8}	$Z_{8,0}$	x^8	$3\sqrt{2}$	$r^8\sin 8\theta$	$-8x^7y+56x^5y^3-56x^3y^5+8xy^7$	
37	8	–6	1	Z_8^{-6}	$Z_{8,1}$	x^7y	$3\sqrt{2}$	$(8r^8-7r^6)\sin 6\theta$	$-42x^5y+140x^3y^3-42xy^5+48x^7y-112x^5y^3-112x^3y^5+48xy^7$	
38	8	–4	2	Z_8^{-4}	$Z_{8,2}$	x^6y^2	$3\sqrt{2}$	$(28r^8-42r^6+15r^4)\sin 4\theta$	$-60x^3y+60xy^3+168x^5y-168xy^5-112x^7y-112x^5y^3+112x^3y^5+112xy^7$	
39	8	–2	3	Z_8^{-2}	$Z_{8,3}$	x^5y^3	$3\sqrt{2}$	$(56r^8-105r^6+60r^4-10r^2)\sin 2\theta$	$-20xy+120x^3y+120xy^3-210x^5y-420x^3y^3-210xy^5-112x^7y+336x^5y^3+336x^3y^5+112xy^7$	
40	8	0	4	Z_8^0	$Z_{8,4}$	x^4y^4	3	$70r^8-140r^6+90r^4-20r^2+1$	$1-20x^2-20y^2+90x^4+180x^2y^2+90y^4-140x^6-420x^4y^2-420x^2y^4-140y^6+70x^8+280x^6y^2+420x^4y^4+280x^2y^6+70y^8$	
41	8	2	5	Z_8^2	$Z_{8,5}$	x^3y^5	$3\sqrt{2}$	$(56r^8-105r^6+60r^4-10r^2)\cos 2\theta$	$10x^2-10y^2-60x^4+105x^4y^2-105x^2y^4+60y^4+105x^6-105y^6-56x^8-112x^6y^2+112x^2y^6+56y^8$	
42	8	4	6	Z_8^4	$Z_{8,6}$	x^2y^6	$3\sqrt{2}$	$(28r^8-42r^6+15r^4)\cos 4\theta$	$15x^4-90x^2y^2+15y^4-42x^6+210x^4y^2+210x^2y^4-42y^6+28x^8-112x^6y^2-280x^4y^4-112x^2y^6+28y^8$	
43	8	6	7	Z_8^6	$Z_{8,7}$	xy^7	$3\sqrt{2}$	$(8r^8-7r^6)\cos 6\theta$	$7x^6-105x^4y^2+105x^2y^4-7y^6-8x^8+112x^6y^2-112x^2y^6+8y^8$	
44	8	8	8	Z_8^8	$Z_{8,8}$	y^8	$3\sqrt{2}$	$r^8\cos 8\theta$	$x^8-28x^6y^2+70x^4y^4-28x^2y^6+y^8$	
45	9	–9	0	Z_9^{-9}	$Z_{9,0}$	x^9	$2\sqrt{5}$	$r^9\sin 9\theta$	$x^9-36x^7y^2+126x^5y^4-84x^3y^6+9xy^8$	
46	9	–7	1	Z_9^{-7}	$Z_{9,1}$	x^8y	$2\sqrt{5}$	$(9r^9-8r^7)\sin 7\theta$	$8x^7-168x^5y^2+280x^3y^4-56xy^6-9x^9+180x^7y^2-126x^5y^4-252x^3y^6+63xy^8$	
47	9	–5	2	Z_9^{-5}	$Z_{9,2}$	x^7y^2	$2\sqrt{5}$	$(36r^9-56r^7+21r^5)\sin 5\theta$	$21x^5-210x^3y^2+105xy^4-56x^7+504x^5y^2+280x^3y^4-280xy^6+36x^9-288x^7y^2-504x^5y^4+180xy^8$	
48	9	–3	3	Z_9^{-3}	$Z_{9,3}$	x^6y^3	$2\sqrt{5}$	$(84r^9-168r^7+105r^5-20r^3)\sin 3\theta$	$20x^3-60xy^2-105x^5+210x^3y^2+315xy^4+168x^7-168x^5y^2-840x^3y^4-504xy^6-84x^9+504x^5y^4+672x^3y^6+252xy^8$	
49	9	–1	4	Z_9^{-1}	$Z_{9,4}$	x^5y^4	$2\sqrt{5}$	$(126r^9-280r^7+210r^5-60r^3+5r)\sin\theta$	$5x-60x^3-60xy^2+210x^5+420x^3y^2+210xy^4-280x^7-840x^5y^2-840x^3y^4-280xy^6+126x^9+504x^7y^2+756x^5y^4+504x^3y^6+126xy^8$	
50	9	1	5	Z_9^1	$Z_{9,5}$	x^4y^5	$2\sqrt{5}$	$(126r^9-280r^7+210r^5-60r^3+5r)\cos\theta$	$5y-60y^3-60x^2y+210y^5+420x^2y^3+210x^4y-280y^7-840x^2y^5-840x^4y^3-280x^6y+126y^9+504x^2y^7+756x^4y^5+504x^6y^3+126x^8y$	
51	9	3	6	Z_9^3	$Z_{9,6}$	x^3y^6	$2\sqrt{5}$	$(84r^9-168r^7+105r^5-20r^3)\cos 3\theta$	$-20y^3+60x^2y+105y^5-210x^2y^3-315x^4y-168y^7+168x^2y^5+840x^4y^3+504x^6y+84y^9-504x^4y^5-672x^6y^3-252x^8y$	
52	9	5	7	Z_9^5	$Z_{9,7}$	x^2y^7	$2\sqrt{5}$	$(36r^9-56r^7+21r^5)\cos 5\theta$	$21y^5-210x^2y^3+105x^4y-56y^7+504x^2y^5+280x^4y^3-280x^6y+36y^9-288x^2y^7-504x^4y^5+180x^8y$	
53	9	7	8	Z_9^7	$Z_{9,8}$	xy^8	$2\sqrt{5}$	$(9r^9-8r^7)\cos 7\theta$	$-8y^7+168x^2y^5-280x^4y^3+56x^6y+9y^9-180x^2y^7+126x^4y^5+252x^6y^3-63x^8y$	
54	9	9	9	Z_9^9	$Z_{9,9}$	y^9	$2\sqrt{5}$	$r^9\cos 9\theta$	$y^9-36x^2y^7+126x^4y^5-84x^6y^3+9x^8y$	
55	10	–10	0	Z_{10}^{-10}	$Z_{10,0}$	x^{10}	$\sqrt{22}$	$r^{10}\sin 10\theta$	$10x^9y-120x^7y^3+252x^5y^5-120x^3y^7+10xy^9$	
56	10	–8	1	Z_{10}^{-8}	$Z_{10,1}$	x^9y	$\sqrt{22}$	$(10r^{10}-9r^8)\sin 8\theta$	$72x^7y-504x^5y^3+504x^3y^5-72xy^7-80x^9y+480x^7y^3-480x^3y^7+80xy^9$	
57	10	–6	2	Z_{10}^{-6}	$Z_{10,2}$	x^8y^2	$\sqrt{22}$	$(45r^{10}-72r^8+28r^6)\sin 6\theta$	$270x^9y-360x^7y^3-1260x^5y^5-360x^3y^7+270xy^9-432x^7y+1008x^5y^3+1008x^3y^5-432xy^7+168x^5y-560x^3y^3+168xy^5$	
58	10	–4	3	Z_{10}^{-4}	$Z_{10,3}$	x^7y^3	$\sqrt{22}$	$(120r^{10}-252r^8+168r^6-35r^4)\sin 4\theta$	$140x^3y-140xy^3-672x^5y+672xy^5+1008x^7y+1008x^5y^3-1008x^3y^5-1008xy^7-480x^9y-960x^7y^3+960x^3y^7+480xy^9$	
59	10	–2	4	Z_{10}^{-2}	$Z_{10,4}$	x^6y^4	$\sqrt{22}$	$(210r^{10}-504r^8+420r^6-140r^4+15r^2)\sin 2\theta$	$30xy-280x^3y-280xy^3+840x^5y+1680x^3y^3+840xy^5-1008x^7y-3024x^5y^3-3024x^3y^5-1008xy^7+420x^9y+1680x^7y^3+2520x^5y^5+1680x^3y^7+420xy^9$	
60	10	0	5	Z_{10}^0	$Z_{10,5}$	x^5y^5	$\sqrt{11}$	$252r^{10}-630r^8+560r^6-210r^4+30r^2-1$	$-1+30x^2+30y^2-210x^4-420x^2y^2-210y^4+560x^6+1680x^4y^2+1680x^2y^4+560y^6-630x^8-2520x^6y^2-3780x^4y^4-2520x^2y^6-630y^8+252x^{10}+1260x^8y^2+2520x^6y^4+2520x^4y^6+1260x^2y^8+252y^{10}$	
61	10	2	6	Z_{10}^2	$Z_{10,6}$	x^4y^6	$\sqrt{22}$	$(210r^{10}-504r^8+420r^6-140r^4+15r^2)\cos 2\theta$	$-15x^2+15y^2+140x^4-140y^4-420x^6-420x^4y^2+420x^2y^4+420y^6+504x^8+1008x^6y^2-1008x^2y^6-504y^8-210x^{10}-630x^8y^2-420x^6y^4+420x^4y^6+630x^2y^8+210y^{10}$	
62	10	4	7	Z_{10}^4	$Z_{10,7}$	x^3y^7	$\sqrt{22}$	$(120r^{10}-252r^8+168r^6-35r^4)\cos 4\theta$	$-35x^4+210x^2y^2-35y^4+168x^6-840x^4y^2-840x^2y^4+168y^6-252x^8+1008x^6y^2+2520x^4y^4+1008x^2y^6-252y^8+120x^{10}-360x^8y^2-1680x^6y^4-1680x^4y^6-360x^2y^8+120y^{10}$	
63	10	6	8	Z_{10}^6	$Z_{10,8}$	x^2y^8	$\sqrt{22}$	$(45r^{10}-72r^8+28r^6)\cos 6\theta$	$-28x^6+420x^4y^2-420x^2y^4+28y^6+72x^8-1008x^6y^2+1008x^2y^6-72y^8-45x^{10}+585x^8y^2+630x^6y^4-630x^4y^6-585x^2y^8+45y^{10}$	
64	10	8	9	Z_{10}^8	$Z_{10,9}$	xy^9	$\sqrt{22}$	$(10r^{10}-9r^8)\cos 8\theta$	$-9x^8+252x^6y^2-630x^4y^4+252x^2y^6-9y^8+10x^{10}-270x^8y^2+420x^6y^4+420x^4y^6-270x^2y^8+10y^{10}$	
65	10	10	10	Z_{10}^{10}	$Z_{10,10}$	y^{10}	$\sqrt{22}$	$r^{10}\cos 10\theta$	$-x^{10}+45x^8y^2-210x^6y^4+210x^4y^6-45x^2y^8+y^{10}$	

The circle polynomials of Zernike are defined over the unit disk D_2. In polar coordinates $0 \le r \le 1$, $0 \le \theta < 2\pi$; in Cartesian coordinates $x^2 + y^2 \le 1$

Connection between the radial polynomials $R_n^m(r)$ and the circle polynomials $Z_n^m(r,\theta)$: The complex combination $Z_n^m(r,\theta) \pm iZ_n^{-m}(r,\theta) = V_n^{-m}(r\cos\theta, r\sin\theta) = R_n^m(r)e^{\pm im\theta}$ leads to $Z_n^m(r,\theta) = R_n^m(r)\cos m\theta$ and $Z_n^{-m}(r,\theta) = R_n^m(r)\sin m\theta$

Polar recursion relationship: $R_n^m(r) = \sum_{i=0}^{m}(-1)^i\frac{(n-s)!}{i!(n-m-i)!(m-i)!}r^{n-2i}$, Cartesian recursion: $Z_n^m(x,y) = \sum_{i=0}^{q}\sum_{j=0}^{M}\sum_{k=0}^{M-j}(-1)^{i+j}\binom{n-2M}{2i+p}\binom{M-j}{k}\frac{(n-j)!}{j!(M-j)!(n-M-j)!}x^{\xi}y^{\eta}$ where $d = n-2m$, $s = \mathrm{sgn}(d)$, $M = \frac{n}{2} - \left|m - \frac{n}{2}\right|$, $p = \frac{|s|}{2}(s+1)$, $q = (d - s\,Mod[n,2])\frac{s}{2}$, $\xi = 2(i+k)+p$, $\eta = n-2(i+j+k)-p$

Chirality of coordinate systems



$(\hat{x} \times \hat{y}) \cdot \hat{z} = -1$ $\qquad$ $(\hat{x} \times \hat{y}) \cdot \hat{z} = 1$

Zernike defined his polynomials in a left-handed coordinate system as shown above on the left. The chirality, or handedness, of a coordinate system is defined by the triple product $(\hat{x} \times \hat{y}) \cdot \hat{z}$. To toggle the functions between the two chirality states use the polar transformation $\pi/2 - \theta' \to \theta$ or $(y'\ x') \to (x\ y)$. This mimics the effect of reflection by a planar mirror. To toggle the Zernike amplitude vectors, apply the reflection matrix $R = \delta_{p_{j,2}}^{p_{i,1}}\delta_{p_{i,2}}^{p_{j,1}}$ to the Taylor amplitude vector. That is, $b' = tz \cdot R \cdot fz \cdot b$. The transformation matrix satisifes the reflection property $R \cdot R = 1$. This chart uses the left-handed convention of Malacara.

Parity of the Zernike polynomials

The angular parity of the Zernike polynomials is determined by the sign of the angular frequency m: $Z_n^{\pm m}(r,-\theta) = \pm Z_n^{\pm m}(r,\theta)$; when $m \ge 0$, the functions are even, when $m < 0$, the functions are odd. In lieu of resolving a wavefront into separate amplitudes for $\pm m$, consider the two terms to describe a state of mixed parity and report one amplitude and α, the angle which defines how the two definite parity states are mixed. The mixing angle is computed via $\cos\alpha_n^M = b_n^m/h$ where $h = \sqrt{(b_n^m)^2 + (b_n^{-m})^2}$ and $M = |m|$.

How Zernike amplitudes transform under arbitrary rotation

The change in the Zernike amplitudes is easy to compute when the wavefront is rotated by an arbitrary angle θ. The rotation is applied to the mixed parity states, not to the definite parity states. The mixing angle is rotated using $\alpha_n'^M = \alpha_n^M + M\theta$. Clearly there is no need to rotate the rotationally invariant states $Z_{2n,n}(x_1, x_2)$.

How Zernike amplitudes transform under aperture reduction

Computing the Zernike amplitudes for a reduced concentric aperture is trivial. In the Taylor basis, this operation is a rescaling of the coordinate axes. Consider the reduced aperture to be smaller by a multiplicative factor $f\ (0 \le f \le 1)$. The $n+1$ amplitudes of order n are multiplied by f^n which is accomplished by multiplying the Taylor amplitudes by a $t \times t$ *matrix* F whose elements are given by $F_{i,j} = f^{g_{i,1}}\delta_j^i$. So given a vector b of Zernike amplitudes, the amplitudes b' for the reduced aperture are $b' = tz \cdot F \cdot fz \cdot b$.

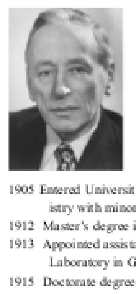
Computing wavefront error using the Zernike amplitudes

Given a vector of Zernike amplitudes b representing a dth order expansion, one can compute σ^2 the wavefront error (or wavefront variance) using

$$\sigma^2 = \overline{\psi^2} - \bar{\psi}^2 = \sum_{i=1}^{d}\sum_{j=0}^{i}\frac{1+\delta_\mu^{n/2}}{2(n+1)}b_{n,\mu}^2.$$

There is no contribution from the piston term since this is the mean value $\bar{\psi}$. For example, in a second order expansion $\sigma^2 = \frac{1}{4}(b_{1,0}^2 + b_{1,1}^2) + \frac{1}{6}(b_{2,0}^2 + 2b_{2,1}^2 + b_{2,2}^2)$. The polynomials can be scaled by the factors in the *RMS* column to simplify the wavefront variance to $\sigma^2 = b_{1,0}^2 + b_{1,1}^2 + b_{2,0}^2 + b_{2,1}^2 + b_{2,2}^2$.

Prof. Dr. Frederick (Frits) Zernike
b. Amsterdam, 7/16/1888; d. 3/10/1966



1946 Appointed member: Royal Dutch Academy of Sciences
1952 Rumford medal: phase contrast microscopy
1953 Nobel prize: phase contrast microscopy
1958 Elected as Foreign Member Royal Society of London

1905 Entered University of Amsterdam majoring in chemistry with minors in physics and math.
1912 Master's degree in chemistry, *cum laude*
1913 Appointed assistant by the head of the Astronomical Laboratory in Groningen University
1915 Doctorate degree in chemistry, *cum laude*
1915 Begins teaching in physics department at Groningen
1920 Full professor of theoretical physics at Groningen

Lasting impacts: 1) development and application of phase contrast and 2) concept of complex coefficients in partial coherence

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The radial polynomials of Zernike: the foundation for the disk polynomials

Symbol: $R_n^m(r)$ or $R_{n,\mu}(r)$; where $m = 2\mu - n$ and $\mu = \frac{n+m}{2}$. *E.g.*, $R_4^0(r) = R_{4,2}(r) = 6r^4 - 6r^2 + 1$

Interval: [0, 1] *Weight*: 1 *Inequality*: $|R(r)| \le 1$ *Standardization*: $R(1) = 1$ *Norm*: $1/(2n+2)$

Differential equation: $r(1-r^2)R'' + (1-3r^2)R' + \left(n(n+2)r - \frac{m^2}{r}\right)R = 0$ where $R = R_n^m(r)$

Explicit expression: $R_n^m(r) = \sum_{l=0}^{(n-m)/2}\frac{(-1)^l(n-l)!}{l!\left[\frac{1}{2}(n+m)-l\right]!\left[\frac{1}{2}(n-m)-l\right]!}r^{n-2l}$

Recurrence relations: $R_n^m(r) = \frac{1}{2(n+1)r}[(n+m+2)R_{n+1}^{m+1}(r) + (n-m)R_{n-1}^{m+1}(r)]$

$$R_{n+2}^m(r) = \frac{n+2}{(n+2)^2 - m^2}\left\{\left[4(n+1)r^2 - \frac{(n+m)^2}{n} - \frac{(n-m+2)^2}{n+2}\right]R_n^m(r) - \frac{n^2-m^2}{n}R_{n-2}^m(r)\right\}$$

$$R_n^m(r) + R_n^{m+2}(r) = \frac{1}{n+1}\frac{d[R_{n+1}^{m+1}(r) - R_{n-1}^{m+1}(r)]}{dr}$$

Relationships to other special functions:

Jacobi polynomials: $R_n^m(r) = (-1)^{\frac{1}{2}(n-m)}r^m P_{\frac{1}{2}(n-m)}^{(m,0)}(1-r^2)$ where $P_n^{(\alpha,\beta)}(x) = \frac{1}{2}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^n(x+1)^m$

Hypergeometric functions: $R_n^m(r) = (-1)^{\frac{n-m}{2}}\binom{\frac{1}{2}(n+m)}{m}r^m{}_2F_1\left(\frac{n+m+2}{2}, -\frac{n-m}{2}, m+1, r^2\right)$ where ${}_2F_1(a,b;c;x) \equiv \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$

Bessel function of the first kind: $\int_0^1 R_n^m(r)J_m(xr)r\,dr = (-1)^{\frac{1}{2}(n-m)}\frac{J_{n+1}(x)}{x}$ where $J_m(x) = \sum_{l=0}^{\infty}\frac{(-1)^l}{2^{2l+m}l!(m+1)!}x^{2l+m}$

Legendre polynomials $P_n(r) = 2^{-n}\sum_{i=0}^{n/2}(-1)^i\binom{n}{i}\binom{2n-2i}{n}r^{n-2i}$ resemble $R_n^0(r) = \sum_{i=0}^{n/2}(-1)^i\binom{n/2}{i}\binom{n-i}{n/2}r^{n-2i}$ where n is even

These special functions can be generated by performing a Gram-Schmidt orthogonalization of a power series

Functions	*Series*	*Interval*	*Weight*	*Norm*
Legendre	$\{1, r, r^2, r^3, \ldots\}$	$-1 \le r \le 1$	1	$2/(2n+1)$
Shifted Legendre	"	$0 \le r \le 1$	1	$1/(2n+1)$
Chebyshev I	"	$-1 \le r \le 1$	$(1-x^2)^{-1/2}$	$\pi/(2-\delta_0^n)$
Shifted Chebyshev I	"	$0 \le r \le 1$	$[x(1-x^2)]^{-1/2}$	$\pi/(2-\delta_0^n)$
Chebyshev II	"	$-1 \le r \le 1$	$(1-x^2)^{1/2}$	$\pi/2$
Associated Laguerre	"	$0 \le r < \infty$	$r^k e^{-r}$	$(n+k)!/n!$
Hermite	"	$-\infty < r < \infty$	e^{-r^2}	$2^n\sqrt{\pi}n!$
Zernike radial	$\{r^m, r^{m+2}, r^{m+4}, \ldots\}$	$0 \le r \le 1$	r	$1/(2n+2)$

Important properties of the circle polynomials of Zernike

1. *Completeness*: $\lim_{d\to\infty}\int_{D_2}\left[\psi(x_1,x_2) - \sum_{n=0}^{d}\sum_{\mu=0}^{n}b_{n,\mu}Z_{n,\mu}(x_1,x_2)\right]dx^2 = 0$

   The polynomial approximation converges uniformly to the input function (Weierstrass theorem).

2. *Orthogonality*: $\int_{D_2}Z_{n,\mu}(x_1,x_2)Z_{n',\mu'}(x_1,x_2)dx^2 = \frac{1+\delta_\mu^{n/2}}{2(n+1)}\pi\delta_n^{n'}\delta_\mu^{\mu'}$. The amplitudes do not change as the order of the fit is increased.

   $\sum_{i=1}^{N}\sum_{j=1}^{N}Z_{n,\mu}(x_{1_i},x_{2_j})Z_{n',\mu'}(x_{1_i},x_{2_j})\frac{\pi}{N^2} \ne \int_{D_2}Z_{n,\mu}(x_1,x_2)Z_{n',\mu'}(x_1,x_2)dx^2$. Discretized functions are **not** orthogonal.

3. *The Cartesian Zernike polynomials are exactly equivalent to a Taylor series order by order.*

   Given a set of Zernike amplitudes $b_{n,\mu}$ there is an exact affine transformation to and from the Taylor amplitudes $a_{n,\mu}$.
   If fz is the transformation from the Zernike basis to the Taylor basis and tz is the transformation from the Taylor basis to the Zernike basis,

   $a = fz \cdot b$, $b = tz \cdot a$. *E.g.*, the transformation matrices for second order are $fz = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$, $tz = \begin{bmatrix} 1 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & -1/2 & 0 & 1/2 \end{bmatrix}$

   The Zernike functions are replaced by vectors in the Taylor basis. The Taylor series $(1, x, y, x^2, xy, y^2, \ldots)$ generalizes to $x^{p_{i,1}}y^{p_{i,2}}$ with the vector of exponential powers $p = ((0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots)$.

Addressing and indexing

1. If the collection of Zernike coefficient vectors is z then $z[n]$ is the coefficient vectors for order n; $z[n][m]$ is the coefficient vectors for the mth function of order n; $z[n][m][k]$ is the coefficient for the kth term in the mth function of order n.
2. A coefficient vector for order n has $t = (n+1)(n+2)/2$ terms; a coefficient vector with t terms represents order $n = \frac{1}{2}(\sqrt{1+8t} - 3)$.
3. The index number of the amplitude for $Z_{n,\mu}(x_1,x_2)$ is $j = n(n+1)/2 + \mu + 1$; the inverse problem is solved by a lookup vector $q = ((0,0), (1,0), (1,1), (2,0), (2,1), (2,2), \ldots)$ which provides the order n and cardinal number m.
4. The order is given by n; the frequency is given by m; the ordinal number is μ.

Example: Primary aberrations through a pupil of radius r:

Taylor basis: $sphere = \frac{a_{2,0} + a_{2,2}}{r^2} + cyl$, $cyl = \frac{2}{r^2}\sqrt{a_{2,0}^2 + a_{2,1}^2 - 2a_{2,0}a_{2,1} + a_{2,2}^2}$, $axis = \frac{1}{2}\mathrm{atan}\left(\frac{a_{2,1}}{a_{2,2} - a_{2,0}}\right)$

Zernike basis: $sphere = \frac{4}{r^2}b_{21} + cyl$, $cyl = \frac{4}{r^2}\sqrt{b_{20}^2 + b_{22}^2}$, $axis = \frac{1}{2}\mathrm{atan}\left(\frac{b_{20}}{b_{22}}\right)$